Cartesian graphs and co-variational thinking in early algebra

Luis Radford

Cartesian graphs and co-variational thinking in early algebra

Luis Radford

Laurentian University, École d'éducation, Sudbury, Canada; Lradford@laurentian.ca

The aim of this article is to contribute to the existing body of research on the understanding of graphs by young students. To this end, I draw on the theory of objectification and focus on the processes of conceptualizing co-variation between continuous variables when Grade 6 students, aged 11–12, interpret a given graph. The context is a dynamic one, namely, the interpretation of a Cartesian graph representing the movement of a body. The findings suggest that, initially, based on perceptual understandings, the students conceptualize the graph as a map, without co-variation being a central aspect of their analyses. Furthermore, the students' perceptual understandings give rise to some contradictions. However, these are gradually resolved as the teacher engages with the students in exploring the graph and new cultural-historical understandings of co-variation emerge.

Keywords: Motion graphs, theory of objectification, co-variational thinking

Introduction

Although research on early algebra has produced a wealth of results in recent years, research on young students' understanding of Cartesian graphs has not received much attention. Indeed, in their recent literature review, Cañadas et al. (2024) note that studies investigating how elementary students construct, use, or read graphs are almost nonexistent. Yet, graphs are known to provide rich functional contexts for learning about variables and their co-variation (Blanton et al., 2018; Kaput, 2008; Kieran, 2016; Robert, 2024). When studied, Cartesian graphs have often been featured in connection with a new functional mode of representing *discrete* data in x/y tables—see, e.g., Cañadas et al. (2024), Carraher et al. (2008), and Tierney and Monk (2008). In this case, graphs are constructed from a functional relation between x and y variables (e.g., $y = 2x+1$), instances of which lead to points (x, y) in the Cartesian plane. In this paper, I focus on a different type of Cartesian graph: a given Cartesian graph representing the *motion* of a body. Understanding such a graph requires *conceptualizing* the co-variation between the inherent *continuous* variables of time and space. Inspired by Carlson et al. (2002), I consider co-variational thinking as the embodied, semiotic, and material cognitive activity "involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). The research question I seek to answer is how students come to conceptualize the co-variation of two variables—time and distance—in a given Cartesian graph.

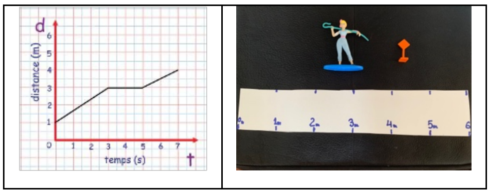
Theoretical framework

The theoretical framework followed in this paper is based on the Vygotskian theory of objectification (TO) (Radford, 2021). In accordance with the TO, the students' conceptualization is part of a sense-making process: the *process of objectification* (p. 86); that is, the social process in which teachers *and* students engage in sensuous and material activity as they proceed to encounter, examine, and explore cultural-historical mathematical meanings—in the case of this paper, mathematical meanings related to Cartesian graphs. This teacher-and-students' sense-making process is a creative, imaginative, affective one entrenched in classroom teaching-learning activity. It finds support and expression in language, signs, artefacts, interactions, gestures, and the body more generally.

Methods

The method to investigate the research question is the dialectical materialist one proposed in the TO (Radford, 2015). This means that *the methodological explanatory category* is the *teaching-learning activity* that anchors the various multi-modal and material connections that teacher and students create as they produce ideas to explore and understand the cultural-historical meaning of Cartesian graphs. The teaching-learning activity is designed in a way that allows the students to work in small groups of three or four and encourages the students to work together in the production of what in the TO is called *une oeuvre commune* (a common artistic work): a collectively built idea about how—in the case of this paper—to interpret graphs. We use four or five video cameras and make transcriptions that are then subjected to multi-modal analyses (Radford, 2021). The data presented below come from a recorded lesson conducted in a Grade 6 class (11–12-year-old students) in a French public school in Sudbury, Canada. In this paper I focus on one of the small groups (chosen for its paradigmatic nature)—a kind of case study. Case studies are limited in terms of the generality of their results. However, they allow in-depth explorations of the research problem. The lesson was preceded by a lesson in which the students had access to a video showing a remotely controlled battery-operated train moving at constant speed in a graduated straight line. They were asked to produce a Cartesian graph of the train's motion. In the lesson featured in this paper (which is an adaptation of a lesson conducted in a Grade 8 class; see Radford, 2009), the graph was given. Here is the problem.

The problem and the first graph interpretations



"Tina is 1 metre from a fountain. She walks on a straight-line path. The graph represents her walk. Give a precise description of Tina's walk." Unlike the lesson in Grade 8, the teacher equipped each group with a kit containing a figurine representing Tina, an object representing the fountain, and a flat, graduated cardboard path to enact Tina's walk. I focus here on one of the groups of three students: D., L., and B.—two girls and a boy, respectively (see Fig. 1.1 below) (Fig m.n refers to pic n, from left to right, in Figure m).

The first interpretation:

D: Huh! Tina starts here (*pointing at (0, 1)*). Then the fountain is here (*pointing at (0, 0)) (Fig. 1.1)*).

L: Yes, but how do we find the time?

D: Time is here. (*She slides her finger along the t-axis*) (*Fig. 1.2*) ... So, from 1 (*she points at (0, 1)*) to here (*she points at (3, 3)*), she did 3s (*Fig. 1.3*). Then, from there (*she points at (3, 3), Fig. 1.4*) to there (*she points at (5, 3), Fig 1.5*), she did 5s .... And then, from here (*she points at (5, 3)*) to there (*she points at (7, 4)*), *Fig. 1.6*) (*she lowers the pen vertically from (7, 4) to (7, 0)*), she did 7s.

Figure 1. Pic 1, from left to right, B., L., and D. The students' first interpretation

In this episode, D. offers a first explanation of the graph based on an interpretation of it as a *map*. She places the figurines of the fountain and Tina at points 0 and 1 of the cardboard, respectively. Then, the attention moves to the graph. The explanation is not clear to L: "You have to explain, I don't understand." D. responds trying to make more explicit her understanding of the graph as a map:

D: Since she came here (*she slides the pen from (0, 1) to (3, 3)*), she went 3m and 3s. She arrived here (*points at (5, 3)*) … 3m and 5s; then, she went straight (*points to graph horizontal segment*).

L: Ok, Ok! Wait! Wait, 1m, it's 7s.

D: No! ... From here (*points at (0, 1)) to there (points at (3, 3)),* it's 3m and 3s. Also, from here (*points at (3,3)*) to here (*points at (5,3)*) it's 5s.

L: Ahhh! So, what's 7s?

D: From here (*points at (5, 3)*) to there (*points at (7, 4)*), it's 7s.

In D.'s interpretation, the t-axis is not conceptualized as a Cartesian variable, the values of which are referred to as a starting point. In her eyes, the numbers on the t-axis are seen as labels or stickers that indicate the duration of the segments that comprise Tina's walk.

L. states that she remains uncertain as to the appropriate explanation for Tina's walk. The teacher comes to see the group. The students explain their current interpretation:

L: (*Referring to the first graph segment*) Tina walks … 3s.

T: So, at 3s, where is she?

D: At 3m. Then (*moving her finger along the horizontal segment*), she walks straight ahead 5s …

T: Does she walk straight ahead? (*She moves her finger across the graph's horizontal segment*) How long?

L: The distance? We can count (*counting horizontal lines on the grid*) 1, 2, 3, 4.

T: (*Pointing to the vertical axis, she calls the students' attention to the fact that the vertical axis indicates the distances. Talking to B. to involve him in the conversation*) What do you see here?

B: The actions Tina did … the meters …

The teacher echoes B.'s remark, emphasizing that the distance is measured along the vertical axis (Fig 2.1) and leaves. The students keep discussing.

The second interpretation: About 5 minutes later, B. suggests a new interpretation based on the vertical axis: "I'm confused, (but) I'll explain. Uh, uh, I don't know. But maybe it's like … like, Tina walked 1 (meter). (*He points at (0, 1) on the Cartesian graph; see Fig. 2.2*). She still goes like ... she goes here (*points at (0, 3)*). And then, she goes here (*points at (0, 4)*), and she's here now (*pointing again at (0, 4)*). Maybe it's like this."

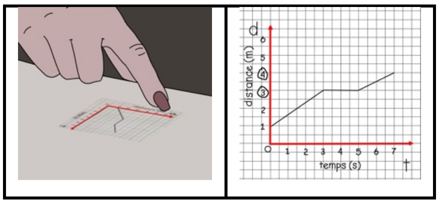


Figure 2. The second interpretation

B.'s interpretation speaks of distance and time in a new manner. Through verbs, B.'s interpretation involves the temporal dimension: Tina *walked*, she *goes here*, but in an unquantified manner. Distance, by contrast is quantified. On the d-axis, B. circles numbers 3 and 4 (see *Fig. 2.2*). It is perhaps the implicit role of time in B.'s interpretation that his teammates find unconvincing; D. and L. go back to the map interpretation of the graph again, as if trying to find something that would so far remain unnoticed.

Third interpretation: L. suggests a third interpretation: "[Tina] started at 1m (*Fig. 3.1*). Then, at three seconds (*she points at 3 on the horizontal axis (Fig. 3.2*) *and slides the pen vertically to point (3, 3); Fig. 3.3*), she continued to 3m (*she points at (0, 3)*) (*Fig. 3.4*); (*repeating*) she continued to 3m (*but now, inspired by what B. said, she moves the pen horizontally to the point (3, 3),*). Then, she continued straight on from 3s to 5s, (*moving the pen along the graph's horizontal segment*), but within 3m (*Fig. 3.5*). Then, she went up ... (*showing the end point of the graph*) to 7s" **(***Fig. 3.6***).**

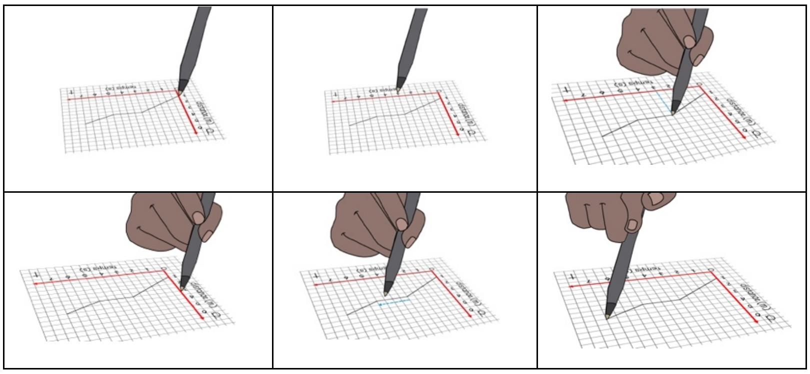


Figure 3. The third interpretation

The interpretation of the horizontal segment persists. But now, *time becomes the main organizer of students' thinking*: at 3s something happened; between 3s and 5s, something else happened, etc. However, the students are not yet satisfied. A few seconds later, L. produces a sharper interpretation: "She started at 1m, ok (*she points at (0, 1); Fig. 4.1*). That's the starting point; and now, at 3s, she was already at 3m (*places the pen between (3, 0) and (3, 3)*; *Fig. 4.2)*. Then, pointing to the horizontal segment, she continued straight on, right? Then at 5s, she started to climb to get to 4m, and at 4m, she is at 7s. Do you understand?"

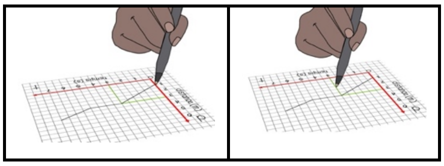


Figure 4. The third interpretation refined

Now, time and distance become related in what we can recognize as a nascent functional interpretation of the graph. Time appears for the first time as a variable; instantiations of the variable (e.g., "at 3 seconds," "at 5 seconds") are put into correspondence with instantiations of the distance seen also as a variable. Yet, the nascent functional interpretation is still haunted by the idea of the graph as a map: the horizontal segment keeps being seen as Tina walking straight on; and in the last segment, Tina is said to climb to 4m. The students start writing their interpretation: "In the beginning, Tina was at 1m and 0s. Then, she walked 3m at 3s. She is at 3m and 3s."

Dealing with the contradictions

When the teacher comes back to see the group, D. offers an explanation that makes the inherent contradictions more visible. D explains: Tina is at "3m and 3s (*she points at (3, 3)*) … Then in the middle of her walk (*she points at (4, 3))* she's still at 3m (*pointing to the horizontal segment*), but she's done 4s (*pointing at (4, 0)*). Then she continued to walk at 3m" (*She moves her pen from left to right on the graph's horizontal segment to indicate Tina's motion*). The contradiction is this: between 3 and 5 seconds, the students imagine that Tina keeps walking while still being at d=3. The teacher encourages the students to keep thinking: "I want you to tell me what happened here (*she points to the horizontal segment*). What do you notice here? What can we add to [the description of] the walk?"

Vygotsky suggested a systemic view of concepts: a concept is formed of dynamic connections between its parts (internal connections) and connections with other concepts (external connections). What distinguishes the concepts that children form in the course of subjective experiences (e.g., "spontaneous" or "quotidian" concepts) from "scientific" concepts is precisely the *nature* of their constitutive connections. Spontaneous concepts often exhibit a *syncretic perceptual understanding* of phenomena. These concepts are based on perceptual connections that form ad hoc *wholes* or *complexes*. Thinking in terms of these perceptual-based connections is what Vygotsky called "complexive thinking" (Vygotsky, 1987, p. 136). What lends "scientific" concepts their particular nature is the cultural-historical character of their internal and external connections. I want to contend that complexive perceptual-based connections are what lead to the perception of Cartesian graphs as maps. Understanding graphs in a Cartesian contemporary mathematical meaning entails becoming aware of its parts: the meaning of the axis, the identification of variables and their co-variational nature (qualitatively and quantitatively)—among other things. The *movement* from complexive thinking to the Western scientific-mathematical co-variational thinking is not a natural process. In the theoretical framework followed here, the name of the process that makes this movement possible *is teaching-learning activity* (Radford, 2021). In teaching-learning activity, *the teacher works hand in hand with the students*; they collectively generate ideas and novel interpretations of the problem at hand. In this lesson, to help the students deal with the contradictions, the teacher suggested that they use the manipulatives. Initially considered by the students, the manipulatives were ultimately set aside. While manipulatives (or artefacts, more generally) are undoubtedly valuable, they may not always be a stand-alone solution for fostering cultural mathematical thinking. The efficacy of artefacts hinges not only on their intrinsic properties but also on their *integration* and use in the teaching-learning activity. Thus, encouraged by the teacher, the students started placing Tina at 1m from the fountain.

L: She moved forward 2m (*Fig. 5.1*)… at 3s (*The teacher moves Tina to the point marked 3*).

T: Ok, at 3s. Now, what's happening? She's here, isn't she? (*She points at (0, 3) on the graph while pointing to Tina in the manipulative model*). What happens with her walk?

D: She walks straight on, uh … she walks straight ahead in 3m and 5s.

T: She's still at 3m (*see Fig.5.2*), but she moves forward …?

L: (*As saying something obvious*) At 3m she moves forward 2s!

T: (*Inviting the students to reconsider*) Does she move forward?

L: Yes! She moves forward 2s!

T: There is some time elapsing here (*she points at the t-axis*), but the distance (*pointing to the horizontal segment in the graph*), does it increase or decrease? … Tina is here (*Fig. 5.3*) … Time elapses (*pointing to the t-axis; Fig. 5.4*). But how long?

L: 2s …

T: Ok. 2s. And *here*, (*touching Tina (Fig. 5.5) and pointing at (0, 3) (Fig. 5.5)*); look at Tina's distance when she was at 3m (*making a sliding gesture pointing at the horizontal segment while keeping the figurine motionless (Fig. 5.6)*). Did she move?

B: No!

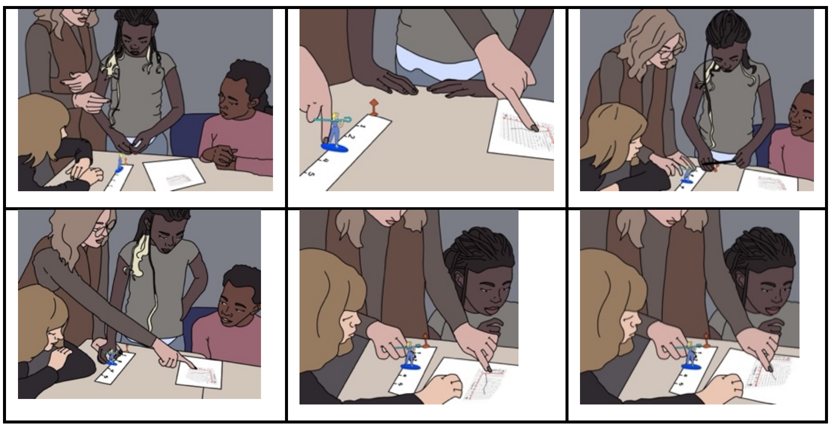


Figure 5. Teacher and students working together

At that moment, the students realized that, in the interpretation of the horizontal segment, changes in one of the variables do not necessarily lead to changes in the other. This new aspect of co-variational thinking had been hindered by the initial percetual sense of the graph as a map. Working hand in hand with the students, the teacher manages to open a new dimension in the conceptualization of Cartesian graphs through a fine-grained link of the enactment of Tina's walk on the cardboard path and the graph. The horizontal graph segment now appears to mean that time elapses while the distance remains the same, thereby overcoming a contradiction that haunted the students' first interpretations of the graph.

The text that the students produced in response to the activity requirements was the following: "In the beginning, Tina did (1m) and (0s). Then she ~~walked~~ advanced (2m at 3s) so she is at (3m and 3s) and 2s pass but she doesn't move so she is at (3m and 5s). Finally, she climbs (1m and 2s) which gives (4m and 7s)." Contradictions still remain. The text is still cast in the rhetorical style of the phenomenological apprehension of the graph as a map: for example, we are told that, in the last segment, Tina climbs. There is still room to move closer to a co-variational description of events. Yet, the ongoing students' encounter with a cultural-historical co-variational understanding of motion graphs paves the way to engage in further reflective mathematical experiences.

Concluding remarks

This paper seeks to contribute to a notorious underrepresented topic in early algebra: the students' understanding of graphs. Cartesian graphs possess an historical epistemological density that may hide its meaning from novice young students, as shown by the example discussed in this article. In our study, which featured a motion graph, drawing on syncretic perceptual understanding of phenomena, the students tended to interpret the given graph as a *map*. D. suggested a first interpretation. In it, thinking about the graph was not organized in terms of Cartesian variables—i.e., variables referred to a certain arbitrary spatial and temporal origin. As the teaching-learning activity unfolded, B. drew on D.'s ideas and on a teacher's hint to formulate a second interpretation where the focus is distance while time remains unqualified. In turn, L. drew on the previous interpretations and formulated a third one, where variables became better identified, although their co-variational nature remained obstructed by the persistent perceptual sense of the graph as a map. At the end of the lesson, drawing on the students' ideas, through a fine-grained link of the enactment of Tina's walk on the cardboard path and the graph, the teacher created room for the students to become conscious of a new meaning of co-variation that discloses the possibility that changes in one of the variables does not necessarily lead to changes in the other. It is this movement of collective ideas that teacher and students form and voice while engaging in teaching-learning activity that we call *l'oeuvre commune*, the collective artistic embodied-intellectual-material work of the participants. The collective idea still requires further refinements, for as noted above, the students' text is still a vivid evocation of the graph as a map walk, rather than a mathematical sign expressing *relations* between variables, the quantities of which "change in relation to each other" (Carlson et al. 2002). "The child's thought [still] must be raised to a higher level for the concept to arise in consciousness" (Vygotsky, 1987, p. 169). This tension between the *perceptual* as the basis of sense that underpins the understanding of the graph as a map, and the *relational* as the basis of sense intended in co-variational thinking is part of the ontogenesis of co-variational thinking. It is not something to avoid, but rather something to help us, educators, to better work with teachers and students.

Acknowledgment

This article is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

References

Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A., Stroud, R., Fonger, N., & Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds, ICME-13 Monographs* (pp. 27–49). Springer.

Cañadas, M., Moreno, A., & Torres, M. (2024). First encounter with constructing graphs in the functional thinking approach to school algebra in 3rd and 4th grades. *ZDM - Mathematics Education*, *56*, 1059–1078. https://doi.org/10.1007/s11858-024-01627-2

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Carraher, D., Schliemann, A., & Schwartz, J. (2008). Early algebra is not the same as algebra early. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Routledge.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), Algebra in the early grades (pp. 5–17). Lawrence Erlbaum Associates.

Kieran, C., Pang, J., Schifter, D., & Ng, S. (2016). *Early algebra. Research into its nature, its learning, its teaching*. Springer Open.

Radford, L. (2009). ‘‘No! He starts walking backwards!’’: Interpreting motion graphs and the question of space, place and distance. *ZDM, 41*, 467–480. https://doi.org/10.1007/s11858-009-0173-9

Radford, L. (2015). Methodological aspects of the theory of objectification. *Perspectivas da Educação Matemática*, *8*, 547–567.

Radford, L. (2021). *The theory of objectification. A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill/Sense. https://doi.org/10.1163/9789004459663

Robert, V. (2024). *La pensée fonctionnelle* [Functional thinking]. Ph. D. Dissertation. Université de Sherbrooke, Canada.

Tierney, C., & Monk, S. (2008). Children’s reasoning about change over time. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 185–200). Routledge.

Vygotsky, L. S. (1987). *Collected works (Vol. 1)*. Plenum.